Frequency Dependence of the Coupling Coefficients and Resonant Frequency Detuning in a NanoPhotonic Waveguide-Cavity System

T. Kamalakis and T. Sphicopoulos

Abstract—Waveguide-cavity interactions may find important applications in future nanophotonic devices. This paper provides a detailed derivation of the evolution equation of the amplitude of a cavity mode coupled to a waveguide, starting from Maxwell’s equations and using the reciprocity relations. The analysis applies to both constant cross-section and periodic waveguides as well. Unlike previous studies, the analysis enables the estimation of the frequency dependence of the coupling coefficients. It is also confirmed that the waveguide-cavity coupling causes a detuning of the resonant frequency of the cavity mode. The detuning is estimated in the case of a photonic crystal waveguide-cavity system and it is shown that it can be significant especially if the structure is intended for filtering applications. The analysis is generalized to the case of a multi-mode or multiple cavities and provides a useful tool in the analysis of devices based on coupled cavities.

Index Terms—

I. INTRODUCTION

Nanophotonic structures are constantly attracting increased attention for the realization of future optical integrated circuits with increased scale of integration. Particular attention is given to Photonic Crystal (PC) structures [1]. For example, PC waveguides provide an efficient means of guiding light by allowing the realization of sharp optical bends [2]. When combined with PC cavities compact optical filter designs are realized [3],[4]. In the non-linear regime, such cavity-waveguide interactions can be used to perform optical switching [5], optical transistor actions [6], and other signal processing functionalities.

In another context, a large chain of coupled cavities [7] can be thought of as a novel type of waveguide where light propagates through evanescent wave coupling from cavity to cavity. This new type of waveguide is called the Coupled Resonator Optical Waveguide (CROW) [8] and has many interesting properties. By appropriately positioning the coupled optical resonators, it is possible to construct sharp, lossless and reflection-less bends throughout the entire CROW band [8]. This is in contrast to bends based on photonic crystal waveguides where reflection-less behavior is observed only at a few optical frequencies [2]. Another important property of a long chain of coupled optical cavities is its ability to drastically slow down the optical field (the slow light concept) [9],[10] which can find important applications in the realization of compact optical delay
lines. When the number of cavities is small, such devices can be used for filtering applications [11]. Other appealing features of the CROW are the low group velocity and large optical field amplitude of the localized modes. These cause an enhancement of non-linear effects [12], which could potentially be useful for all-optical signal processing purposes. In a PC-based CROW (figure 1a), light is fed from a PC waveguide at the CROW input and a second PC waveguide collects the output light. Another structure based on waveguide-cavity interaction is the Side Coupled Integrated Sequence of Resonators (SCISSOR) [13]. The SCISSOR (figure 1b) consists of a chain of coupled cavities side-coupled to a waveguide. Light propagates with small group velocity inside a SCISSOR and nonlinear effects are enhanced. By cascading a CROW and a SCISSOR it is possible to compensate third-order dispersion effects occurring in each separate device and significantly increase the bandwidth on which slow light propagation can take place [14].

CROWs and the SCISSORs are examples of devices based on waveguide/cavity interactions and it is therefore evident that many future optical integrated circuits may rely on waveguide-cavity interactions. These interactions can be analyzed using direct numerical solution of Maxwell’s equations, e.g. Finite Difference Time Domain (FDTD) schemes [15]. Such schemes usually require increased computational time and memory resources, especially for the simulation of long devices. Alternatively one can use Coupled Mode Theory (CMT) [16] to describe the evolution of the amplitude of the cavity and waveguide modes. CMT can furnish a useful physical insight to the waveguide-cavity coupling and can potentially be used to study the influence of perturbations such as the ones caused by fabrication-induced disorder [17],[18]. These perturbations are usually very small position and size variations, which require very fine grid in order to be captured by FDTD and other time or frequency domain grid-based numerical techniques. On the other hand, CMT can handle arbitrarily small structure perturbations.

In the context of the CMT, the time evolution equation for the amplitude $a$ of a single mode cavity is written as [19]:

$$\frac{d\bar{a}}{dt} = -j\omega_0\bar{a} - \frac{1}{\tau}\bar{a} + K_f\bar{a}_f + K_b\bar{a}_b$$

(1)

where $\omega_0$ is the resonant frequency, $\tau$ is the decay rate, $a_f$ and $a_b$ are the amplitudes of the forward and backward propagating mode at the device input and output respectively (figure 2) and $K_f, K_b$ are the coupling coefficients between the cavity mode and the forward or backward propagating waveguide modes, respectively. The bar symbol over the modal amplitudes denotes that they are assumed in the time domain. Equation (1) has been widely used in the literature and has been verified by fitting its results with FDTD calculations. However, there has not been any derivation of (1) starting directly from Maxwell’s equations, especially in the case of photonic crystal structures. In [20], (1) is introduced in a phenomenological way without relating the coupling coefficients to the waveguide modes. In [19], (1) is postulated and then $K_f, K_b$ are derived at resonance from energy conservation arguments. A system of coupled equations for the waveguide-cavity interaction is derived in [21] but is based on
the superposition of waveguide modes over many different propagation constants and consequently (1) does not directly follow from this system. In [20], the inclusion of the waveguide amplitudes is done in a phenomenological manner while in [23],[24] the theory is extended to multimode cavities in an empirical fashion, without involving Maxwell’s equation.

In this paper we present a detailed derivation of (1) directly from Maxwell’s equations using reciprocity relations [25]. This provides a precise account of all the approximations under which the evolution equation (1) holds. The analysis applies for either conventional (constant cross-section) or periodic waveguides (e.g. PC waveguides). The values of $K_f$ and $K_b$ are calculated directly from the reciprocity relations revealing their full frequency-dependent nature. The analysis also predicts a detuning $\Delta \omega_0$ at the frequency of cavity mode oscillations with respect to the resonant frequency $\omega_0$ of the isolated cavity mode due to waveguide-cavity coupling. $\Delta \omega_0$ is expressed in terms of the coupling coefficients of the waveguide and the cavity and is estimated in the case of a photonic crystal cavity/waveguide system. As shown $\Delta \omega_0$ can be significant, especially if the structure is intended for filtering applications. The analysis is also extended to include the interactions of a single mode waveguide with a multi-mode cavity or multiple cavities. This generalization is important since the coupled cavities of both the CROW and the SCISSOR can be treated as either a series of coupled cavities or a single multi-mode cavity, the modes of which can be expanded in terms of the isolated cavity modes while the coefficients of these expansions are known in closed form [26].

II. PRELIMINARY CONSIDERATIONS

Assuming time harmonic fields of the form $e^{j\omega t}$ inside the waveguide-cavity system (figure 2) the total electric and magnetic fields $\mathbf{E}, \mathbf{H}$ are approximately written as a superposition of the isolated waveguide and cavity modes,

$$\mathbf{E}(r; \omega) = a \mathbf{E}_r + a_f \mathbf{E}_f + a_b \mathbf{E}_b$$  \hspace{1cm} (2a)

$$\mathbf{H}(r; \omega) = a \mathbf{H}_r + a_f \mathbf{H}_f + a_b \mathbf{H}_b$$  \hspace{1cm} (2b)

$E_m, H_m$ being the electric and magnetic field variation of the forward ($m=f$), backward ($m=b$) and cavity modes ($m=r$). The cavity mode amplitude $a$ does not depend on the propagation direction $z$, while the amplitudes of the forward and backward propagating modes $a_f$ and $a_b$ are taken $z$-dependent as usual [25]. Note that the field expansion in terms of the isolated modes is valid for weak coupling (which is the case in CROWs and SCISSORs). A similar expansion in terms of the isolated cavity modes has been used in the analysis (without considering the input/output waveguides) of both an infinite [8],[27] and a finite CROW [26] and has been shown to lead to accurate results.

The total electromagnetic field obeys Maxwell’s equations [25]:

$$\nabla \times \mathbf{E} = j\omega \mu \mathbf{H}$$  \hspace{1cm} (3)
\[ \nabla \times \mathbf{H} = -j \omega \varepsilon \mathbf{E} \]  

(4)

In (4), \( \varepsilon = \varepsilon(r) \) is the dielectric constant of the waveguide-cavity system. The isolated waveguide modes obey the following equations:

\[ \nabla \times \mathbf{E}_m = j \omega \mu \mathbf{H}_m \]  

(5)

\[ \nabla \times \mathbf{H}_m = -j \omega \varepsilon_m \mathbf{E}_m \]  

(6)

In (6), \( \varepsilon_w = \varepsilon_w(r) \) is the dielectric constant of the isolated waveguide and the subscript \( m = f \) for the forward or \( m = b \) for the backward propagating mode. In the case of periodic waveguides along the \( z \)-direction, like the one in figure 1a, there exists a vector \( r_a = r, z \) for which \( \varepsilon_w(r) = \varepsilon_w(r + r_a) \). In this case, invoking Bloch’s theorem the modal fields are expressed as [28]:

\[ \mathbf{E}_m(r) = \varepsilon_m(r)e^{\pm j \beta z} \]  

(7)

\[ \mathbf{H}_m(r) = \mathbf{h}_m(r)e^{\pm j \beta z} \]  

(8)

where the exponent \( \pm j \beta z \) in (7)-(8) is \( j \beta z \) for the forward and \( -j \beta z \) for the backward propagating mode respectively. The constant \( \beta \) is the propagation constant of the mode and the vector functions are periodic, i.e. \( \varepsilon_m(r + r_a) = \varepsilon_m(r) \) and \( \mathbf{h}_m(r + r_a) = \mathbf{h}_m(r) \).

Equations (7) and (8) also hold for conventional, constant cross-section waveguides [25], in which case \( r_a \rightarrow 0 \) and the functions \( \varepsilon_m \) and \( \mathbf{h}_m \) depend only on the transverse coordinates \( x \) and \( y \).

The forward and backward modes can be chosen so as to fulfill the following symmetry relations [28]:

\[ \varepsilon_f = \varepsilon_b^*, \quad E_f = E_b^* \]  

(9)

\[ h_f = -h_b^*, \quad H_f = -H_b^* \]  

(10)

The waveguide modes are mutually orthogonal, i.e.:

\[ \int_S \left( E_f^* \times H_b + E_b^* \times H_f^* \right) zdS = 0 \]  

(11)

where \( S \) is a plane normal to the \( z \)-direction. This condition is well known in the case of constant cross-section waveguides [25]. The orthogonality relations for periodic waveguides however, are usually written in terms of volume integrals [28] and involve integration along the \( z \)-direction as well. Hence, applying the orthogonality relations in this way does not allow the direct derivation of simple coupled mode equations for \( a_f \) and \( a_b \) since the \( z \)-dependence can not be separated [29]. However using reciprocity relations, it has been recently shown that (11) holds for the guided modes of periodic waveguides as well [30]. This allows the separation of the \( z \)-dependence and the treatment of periodic waveguides in the same way as in conventional waveguides in the derivation of the coupled mode equations as will be revealed in the next sections.

The waveguide modes are normalized so that
where the right hand side equals +1 for the forward propagating mode \((m=f)\) and –1 for the backward propagating mode \((m=b)\).

The cavity mode is determined from the following equations:

\[
\nabla \times \mathbf{E}_r = j \omega_0 \varepsilon_0 \mu_0 \mathbf{H}_r \tag{13}
\]

\[
\nabla \times \mathbf{H}_r = -j \omega_0 \varepsilon_0 \mathbf{E}_r \tag{14}
\]

In (13)-(14) \(\omega_0\) and \(\varepsilon_c=\varepsilon_c(r)\) are the resonant frequency of the mode and the dielectric constant of the isolated cavity respectively.

The electric field of the cavity mode is taken purely real [31], in which case the magnetic field \(\mathbf{H}_r=j(\omega_0\mu_0)^{-1}\nabla \times \mathbf{E}_r\) is purely imaginary as indicated by (13) and hence,

\[
\mathbf{E}_r^* = \mathbf{E}_r, \quad \mathbf{H}_r^* = -\mathbf{H}_r \tag{15}
\]

The evolution equation (1) will be derived in the following section by applying the reciprocity relation on the vectors \(\mathbf{F}_m=\mathbf{E}_m \times \mathbf{H}_m^* + \mathbf{E}_m^* \times \mathbf{H}_m\) where the index \(m\) is either \(f, b\) or \(r\).

III. EQUATIONS FOR THE WAVEGUIDE MODE AMPLITUDES

In this section we derive the coupled mode equations for the waveguide amplitudes \(a_f(z)\) and \(a_b(z)\). These equations can be used to determine the \(z\)-dependence of these amplitudes and will be used in a later section to simplify the coupled mode equation for the cavity mode amplitude \(a\). Defining the vector function

\[
\mathbf{F}_f = \mathbf{E} \times \mathbf{H}_f + \mathbf{E}_f^* \times \mathbf{H}
\]  

and using the reciprocity relation (A5) with \(\mathbf{E}_1=\mathbf{E}, \mathbf{H}_1=\mathbf{H}, \mathbf{E}_2=\mathbf{E}_f, \mathbf{H}_2=\mathbf{H}_f, \omega_1=\omega_2=\omega, \varepsilon_1=\varepsilon, \varepsilon_2=\varepsilon_w\) one obtains:

\[
\frac{\partial}{\partial z} \int_S \mathbf{F}_f \cdot \mathbf{z} dS = j \omega \int_S (\varepsilon - \varepsilon_w) \mathbf{E}_f^* \cdot \mathbf{E} dS \tag{17}
\]

where referring to figure 2, the surface \(S=\{(x,y,z); -\infty<\chi, \varepsilon<\infty, z=z_1\}\) is any plane perpendicular to the \(z\)-axis. Note that equation (17) is derived by assuming that the electromagnetic field vanishes at the contour \(\partial S\) enclosing \(S\) at infinity. Using (2a) the right hand side of (17) is written as

\[
j \omega \int_S (\varepsilon - \varepsilon_w) \mathbf{E}_f^* \cdot \mathbf{E} dS = j \omega \alpha \int_S (\varepsilon - \varepsilon_w) \mathbf{E}_f^* \cdot \mathbf{E}_f dS + j \omega \alpha \int_S (\varepsilon - \varepsilon_w) \mathbf{E}_f^* \cdot \mathbf{E}_s dS \tag{18}
\]
The second integrand in the right hand side of (18) involves the square of the field of the forward propagating mode weighted by \( \varepsilon - \varepsilon_w \). Since \( \varepsilon - \varepsilon_w \) is non-zero only inside the cavity where \( |E_f|^2 \) has negligible value (due to the weak-coupling assumption), one can neglect this self-coupling contribution. With similar arguments one can neglect the third integral in the right hand side of (18) as well, in which case,

\[
j\omega \int \left( \varepsilon - \varepsilon_w \right) E_f^* \cdot E \, dS \equiv j\omega a \int \left( \varepsilon - \varepsilon_w \right) E_f^* \cdot E, dS
\]  

Using (2a), (11) and (12), the left hand side of (17) is written as

\[
\frac{\partial}{\partial z} \int F \, z \, dS = \frac{\partial a_f}{\partial z} + a \frac{\partial}{\partial z} \int (E_r \times H_f^* + E_f \times H_r) \, z \, dS
\]  

It should be emphasized that the above equation holds for both constant cross-section and periodic waveguides as well since (11) and (12) hold in both cases. The second integral of (20) can be evaluated using the reciprocity relation (A5). The fields \((E_r, H_r)\) and \((E_f, H_f)\) obey Maxwell equation (5)-(6) and (13)-(14) respectively. Setting \(E_1=E_r, H_1=H_r, E_2=E_f, H_2=H_f\) and \(\varepsilon_1=\varepsilon_c, \omega_1=\omega_0, \epsilon_2=\varepsilon_w, \omega_2=\omega\) in (A5) one obtains:

\[
\frac{\partial a_f}{\partial z} = j a \kappa_f
\]  

where the z-dependent coupling coefficient \(\kappa_f\) is

\[
\kappa_f = \int \left( \omega \varepsilon - \omega_0 \varepsilon_c \right) E_f^* \cdot E, dS + \mu \left( \omega - \omega_0 \right) \int H_f^* \cdot H, dS
\]

Equation (21) provides the coupled mode equation for the amplitude of the forward propagating mode. Note that the coupling coefficient \(\kappa_f\) contains a contribution due to magnetic field coupling. This contribution vanishes at resonance, (i.e. at \(\omega=\omega_0\)) in which case \(\kappa_f\) attains the value derived by energy conservation in [19]. In a similar fashion one can derive a coupled mode equation for the backward propagating amplitude

\[
\frac{\partial a_b}{\partial z} = -j a \kappa_b
\]  

where the minus sign is due to the normalization of the backward modes in (12) and the coupling coefficient \(\kappa_b\) is given by

\[
\kappa_b = \int \left( \omega \varepsilon - \omega_0 \varepsilon_c \right) E_b^* \cdot E, dS + \mu \left( \omega - \omega_0 \right) \int H_b^* \cdot H, dS
\]
Using (9)-(10) and (15) one can easily verify that the two coupling coefficients are related through $\kappa_b = \kappa_f^*$. Equations (21) and (23) will be useful in the simplification of the coupled mode equation for the cavity amplitude $a$, which will be derived in the next section.

IV. EQUATION FOR THE CAVITY AMPLITUDE

In this section we derive the evolution equation for the cavity amplitude by applying the reciprocity relations. The derivation of the time domain evolution equation (1) is based on this coupled mode equation. To obtain the latter, we define the vector function

$$ F_j = E \times H^*_j + E^*_j \times H $$  

and using reciprocity relation (A5) we obtain

$$ 0 = j\mu(\omega - \omega_o)\int_V H^*_j \cdot H dV + j\int_V (\varepsilon\omega - \varepsilon_o)E^*_j \cdot E dV $$  

(26)

The volume $V$ is defined as $V=\{(x,y,z); -\infty < x,y < \infty, -l \leq z \leq l\}$ where the value of $l$ is chosen so that the resonator mode $E_r, H_r$ has negligible values at $z=\pm l$. Hence the surface integral at $\partial V$ of (A5) can be ignored. Using the expansion (2a)-(2b), equation (26) is written as

$$ j\int_{-l}^l dz a_j(z)\kappa^*_j(z) + j\int_{-l}^l dz a_b(z)\kappa^*_b(z) + j\kappa a = 0 $$  

(27)

where the cavity mode self-coupling coefficient $\kappa$ is defined as

$$ \kappa = \mu(\omega - \omega_o)\int_V H^*_j \cdot H_j dV + j\int_V (\varepsilon\omega - \varepsilon_o)E^*_j \cdot E_j dV $$  

(28)

Equation (27) is the coupled mode equation for the cavity amplitude. Note that (28) contains the frequency dependence of the coupling coefficient. According to (27) the cavity amplitude is determined by the ensemble of values of the modal amplitudes $a_j(z)$ and $a_b(z)$ throughout the $z$-axis. In the next subsection it will be shown that (27) can be simplified so as to yield an equation relating the cavity amplitude to the amplitudes $a_j(-l)=a_{j0}$ and $a_b(l)=a_{b0}$ of the forward waveguide mode at the device input $z=-l$, and the backward waveguide mode at the device output $z=l$ respectively.

V. SIMPLIFICATION OF THE EQUATION FOR THE CAVITY AMPLITUDE

To simplify the first integral of (27) one can use integration by parts in which case
$$\int_{-l}^{l} a_j(z) \kappa_j^*(z) dz = \int_{-l}^{l} a_j(z) \frac{\partial \Lambda_j^*(z)}{\partial z} dz =$$

$$\Lambda_j^*(l) a_j(l) - \Lambda_j^*(-l) a_j(-l) - \int_{-l}^{l} \frac{\partial a_j(z)}{\partial z} \Lambda_j^*(z) dz$$

(29)

where the function $A_i(z)$ is given from

$$\frac{\partial \Lambda_i(z)}{\partial z} = \kappa_i$$

(30)

Choosing $A_i(z)$ to be zero at $z=l$ one obtains

$$\Lambda_i(z) = \int_{-l}^{l} \kappa_i dz$$

(31)

Using the fact that $A_i(l)=0$ and equation (21), the following equation is derived from (29):

$$\int_{-l}^{l} a_j(z) \kappa_j^*(z) dz = -\Lambda_j^*(-l) a_{j0} - ja \int_{-l}^{l} \kappa_j(z) \Lambda_j^*(z) dz$$

(32)

A similar expression is obtained for the second integral

$$\int_{-l}^{l} a_b(z) \kappa_b^*(z) dz = \Lambda_b^*(l) a_{b0} + ja \int_{-l}^{l} \kappa_b(z) \Lambda_b^*(z) dz$$

(33)

where the function $A_b(z)$ is defined as

$$\Lambda_b(z) = \int_{-l}^{l} \kappa_b dz$$

(34)

Note that since $A_b(-l)=0$ the term involving $a_b(-l)$ vanishes. Using (32) and (33), the coupled mode equation (27) is reduced to

$$j\kappa + K_f a_{f0} + K_b a_{b0} + Ia = 0$$

(35)

where

$$I = \int_{-l}^{l} dz \kappa_j(z) \Lambda_j^*(z) - \int_{-l}^{l} dz \kappa_b \Lambda_b^*(z)$$

(36)

and the coupling coefficients $K_f$ and $K_b$ are defined by

$$K_f \equiv -j\Lambda_j^*(-l) =$$

$$j \int \left( \omega c - \omega_b c_i \right) E_f \cdot E_f^* dV - j \mu \left( \omega - \omega_b \right) \int \left| H_f \cdot H_f^* \right| dV$$

(37)

$$K_b \equiv j\Lambda_b^*(l) =$$

$$j \int \left( \omega c - \omega_b c_i \right) E_b \cdot E_b^* dV - j \mu \left( \omega - \omega_b \right) \int \left| H_b \cdot H_b^* \right| dV$$

(38)
Equation (35) relates the cavity amplitude $a$ to the amplitudes $a_{f0}$ and $a_{b0}$ of the forward and backward waveguide mode at the device input and output respectively, through the coupling coefficients $K_f$ and $K_b$. This is in contrast to (27) where the ensemble of values of $a_f(z)$ and $a_b(z)$ are needed to determine the cavity amplitude. Hence (35) can yield the value of the cavity amplitude from the incident fields. In the next section, equation (35) will be used in order to derive (1) after making some further simplifications assuming that the system operates near resonance ($\omega \approx \omega_0$).

VI. TIME-DOMAIN EVOLUTION EQUATION FOR THE CAVITY AMPLITUDE

Equation (35) can be brought to the time domain by applying the inverse Fourier transform. However the coefficients $\kappa$, $I$, $K_f$ and $K_b$ depend on the frequency $\omega$ and this will in general complicate the evolution equation involving the time derivatives of $a_{f0}$ and $a_{b0}$. Alternatively, one can assume that the system operates near resonance and that the spectrum of the field is located near $\omega \approx \omega_0$. Note however that since the fields are real, there is also a portion of the spectrum located near $\omega \approx -\omega_0$. A real signal $x(t)$ can always be written as $x(t) = x_1(t) + x_2(t)$ where $x_1$ and $x_2$ correspond to the portion of the spectrum located near $\omega \approx \omega_0$ and $\omega \approx -\omega_0$ respectively. Since $x(t)$ is real, $x_1^*(t) = x_2(t)$ and one can deal only with $x_1(t)$ and then recover $x(t)$, by setting $x(t) = 2 \text{Re}\{x_1(t)\}$. For example, although the values of $a$ in (1) are complex, the field amplitude is actually $2 \text{Re}\{a(t)\}$ if the portion of the spectrum located near $\omega \approx -\omega_0$ is taken into account. Alternatively, one could add the complex conjugate of the right hand side in (1).

In the case $\omega \approx \omega_0$ the frequency dependence in $I$, $K_f$ and $K_b$ can be ignored. To show this, $K_f$ and $K_b$ are first expressed in terms of the electric fields alone. This can be done since $E_r$ and $H_r$ can be assumed vanishing at $z=\pm l$ in which case (B2) is written as

$$\mu \int H_r \cdot H^*_f dV = \frac{\alpha}{\omega} \int E_r \cdot E^*_f dV \quad (39)$$

Inserting (39) into (37) one obtains

$$K_f = j \frac{1}{\omega} \int \left( \omega^2 \varepsilon - \omega^2_0 \varepsilon_c \right) E_f \cdot E^*_r dV \quad (40)$$

In a similar fashion one has

$$K_b = j \frac{1}{\omega} \int \left( \omega^2 \varepsilon - \omega^2_0 \varepsilon_c \right) E_b \cdot E^*_r dV \quad (41)$$

Assuming that the system operates near resonance, i.e $\omega \approx \omega_0$

$$K_f \approx j \omega_0 \int \left( \varepsilon - \varepsilon_c \right) E_f \cdot E^*_r dV \quad (42)$$
\[ K_b \approx j\omega_b \int \left( \varepsilon - \varepsilon_c \right) E_b \cdot \mathbf{E}^* dV \] (43)

The self-coupling coefficient can also be simplified by assuming suitable cavity mode normalization. As in the derivation of (19), one can neglect the cavity mode self coupling in the waveguide (i.e. \( \int \varepsilon \left( \varepsilon - \varepsilon_c \right) |E|^2 dV \approx 0 \)) and obtain

\[ \kappa \approx \omega - \omega_0 \] (44)

provided that the cavity mode is normalized to have unit total electromagnetic energy i.e.

\[ \mu \int \mathbf{H}_r^* \cdot \mathbf{H}_r dV + \int \varepsilon \mathbf{E}_r^* \cdot \mathbf{E}_r dV = 1 \] (45)

The real part \( \text{Re}\{I\} \) of \( I \) can also be calculated using a simpler expression. Using the fact that \( \text{Re}\{I\} = (I + I^*)/2 \) and (36), along with the definitions (37)-(38), it is easily shown that

\[ \text{Re}\{I\} = -\frac{|K_f|^2 - |K_b|^2}{2} \] (46)

where we have denoted \( \text{Re}\{I\} \) by \(-1/\tau\). To estimate the imaginary part \( \text{Im}\{I\} \) of \( I \) we first note if the cavity mode is real, then applying (9),(10),(15) and (37)-(38) one obtains from (46),

\[ \left| K_f \right|^2 = \left| K_b \right|^2 = -\text{Re}\{I\} = \frac{1}{\tau} \] (47)

Also using integration by parts

\[ \int_{-l}^{l} \kappa_j \Lambda_j^* dz = \left| K_j \right|^2 - \int_{-l}^{l} \kappa_j^* \Lambda_j dz \]

\[ = \text{Re}\{I\} - \int_{-l}^{l} dz \kappa_j^* \Lambda_j \] (48)

Using (48) and (36) one obtains

\[ \Delta \omega_0 = -\text{Im}\{I\} = \frac{1}{j \int_{-l}^{l} \left( \kappa_j^* \Lambda_j + \kappa_b \Lambda_b^* \right) dz} \] (49)

where we have denoted \( \text{Im}\{I\} \) by \(-\Delta \omega_0\). Using (44),(46) and (49) one can rewrite (35) as

\[ K_f a_{f_0} + K_b a_{b_0} - a \frac{1}{\tau} - j \Delta \omega_0 a + j(\omega - \omega_0) a = 0 \] (50)

where according to (42)-(43) the coefficients \( K_f \) and \( K_b \) are assumed independent of \( \omega \). Applying the inverse Fourier transform and noting that in the time domain, \(-j\omega\) is replaced by the time derivative \( d/dt \), one obtains the following time evolution equation for the cavity amplitude

\[ \frac{d\bar{a}}{dt} = -j \left( \omega_0 + \Delta \omega_0 \right) \bar{a} - \frac{1}{\tau} \bar{a} + K_f \bar{a}_{f_0} + K_b \bar{a}_{b_0} \] (51)
Equation (51) is of the same form as (1) but the frequency of cavity oscillations is detuned $\Delta \omega_0$ away from the isolated cavity resonant frequency $\omega_0$, due to the coupling between the cavity and the waveguide. The value of $\Delta \omega_0$ can be determined from the coupling coefficients involving the field distributions of the modal fields and (49). In the absence of any waveguide coupling $K_f = K_b = \Delta \omega_0 = 0$ and the amplitude of the cavity mode oscillates in time at a frequency $\omega_0$.

If the difference $\omega - \omega_0$ is such that the $\omega$-dependence cannot be ignored then one could use Taylor’s expansion to expand the coupling coefficients in terms of the powers of $\omega - \omega_0$. In the time domain, this will result in having higher order derivatives in equation (1).

VII. ESTIMATION OF THE RESONANT FREQUENCY DETUNING

To estimate the value of $\Delta \omega_0$ we assume a 2D photonic crystal defect cavity coupled to a 2D photonic crystal waveguide as shown in figure 3a. In the 2D case, the electric field can be taken to lie in the $y$-direction (parallel to the rods) $\mathbf{E} = y \mathbf{E}$ and the magnetic field is perpendicular to $y$-direction, i.e. $\mathbf{H} = (H_x, 0, H_z)$. The waveguide is formed by removing a single row of rods from an otherwise rectangular periodic lattice. The center-to-center distance between the rods is $r_a = 0.6 \mu m$ and their radius is $r = 0.18 r_a$.

The rods have refractive index equal to $n_{\text{rods}} = 3.4$ and are surrounded by air ($n_{\text{air}} = 1.0$). The removal of a single rod creates a “defect” mode with resonant frequency which can be calculated using the Plane Wave Expansion (PWE) technique near $f_0 = \omega_0 / 2\pi = 193.8 \text{THz}$ (corresponding to a resonant wavelength of $1.548 \mu m$). Figure 3b illustrates the dielectric constant for the isolated cavity (designated by $\varepsilon_c$ in the previous sections) and figure 3c illustrates the dielectric constant for the isolated waveguide (designated by $\varepsilon_w$). Figure 3d illustrates the difference $\varepsilon - \varepsilon_c$ used in the calculations of the coupling coefficients $\kappa_f$ and $\kappa_b$. The dashed-patterned regions represent the only non-zero regions of $\varepsilon - \varepsilon_c$. These regions lie inside the photonic crystal waveguide.

The cavity mode field distribution, calculated by the plane wave expansion method [28] is shown in figure 4a. The waveguide supports one forward and one backward propagating mode. The square root $|\mathbf{E}_f|$ of the intensity $|\mathbf{E}_f|^2 = |\mathbf{e}_f|^2$ of the forward propagating mode $\mathbf{E}_f$ is depicted in figure 4b. The modal distribution was calculated by the plane wave expansion method in a similar fashion, and as expected is periodic with $z$. Due to the large refractive index contrast of the dielectric rods and the surrounding dielectric (air), the mode is tightly confined inside the waveguide. Note that the waveguide modes and the cavity mode are normalized according to (12) and (45) respectively.

Given the distribution of $\varepsilon - \varepsilon_c$ and the mode distributions of the electromagnetic field of the cavity and the waveguide, one may proceed to evaluate the coupling coefficients $\kappa_f(z)$ and $\kappa_b(z)$ through (22) and (24), respectively, using the modal distributions of figures 4a-4b and numerical integration. The real and imaginary parts of $\kappa_f$ are plotted in figure 5. Due to the symmetry of the
waveguide around the line \( z=0 \), the mode \( \mathbf{E}_f(x,z) = \mathbf{E}_f^*(x,-z) \). The cavity mode also obeys a symmetry relation, \( \mathbf{E}_r(x,z) = \mathbf{E}_r(x,-z) \), with the exception that there is no complex conjugation since the mode is taken purely real. Hence, as shown in figure 5, one has \( \kappa_f(z) = \kappa_f(-z) \). As expected \( \kappa_f \) is non-zero only for values of \( z \) for which \( \varepsilon - \varepsilon_c \) is non-zero as illustrated in figure 3d. Furthermore, the coupling diminishes strongly away from the center of the cavity. As shown in figure 5, the values of the coupling coefficient become practically negligible at \( z = \pm 3r_a = \pm 1.8 \mu\text{m} \), i.e three lattice periods away from the cavity center. This is because the cavity mode decays rapidly and is practically negligible at these distances and thus one can choose \( l = 3r_a \).

The values of the coupling coefficient reveal that the coupling is mainly confined inside the region \( |z| \leq 1.5r_a \), i.e. in the rectangle region indicated in figure 3(a) with dashed lines. Therefore any perturbations to the dielectric constant \( \varepsilon \) (e.g. due to fabrication-induced disorder) affect the waveguide-cavity coupling only if they lie within the shaded region. The values of the function \( \Lambda_f(z) \) can be obtained from (31) and are plotted in figure 6. Note that \( \Lambda_f(z) \) tends to zero as \( z \) increases. This is expected since \( \Lambda_f(z) \) is defined in such a way that \( \Lambda_f(l) = 0 \), where \( l = 3r_a \).

The values of \( \kappa_f(z) \) and \( \Lambda_f(z) \) can be obtained in a similar fashion. The values of the frequency detuning \( \Delta f_0 = \Delta \omega_0/(2\pi) \) from the resonant frequency of the mode of the isolated cavity \( f_0 = 193.8\text{THz} \) are then calculated using (49) and are quoted in table I for three different spacings between the cavity and the waveguide. If a single rod separates the cavity and the waveguide, a somewhat large detuning results, equal to \( \pm 550\text{GHz} \), as the coupling between the cavity and the waveguide is stronger in this case. For a two-rod gap, the detuning is reduced to \( \pm 100\text{GHz} \). Although this is a smaller percentage of the resonant frequency \( f_0 \), it is comparable to the channel spacing used in conventional Wavelength Division Multiplexing (WDM) systems [32]. The frequency detuning further decreases for the three-rod gap to \( \pm 20\text{GHz} \). These values of the frequency detuning once again highlight the sensitivity of small perturbations to the spectral characteristics of nanophotonic devices.

Figure 7, compares the values of the coupling coefficient \( K_f \) computed either with equation (40), which includes the frequency dependence of \( K_f \), and with (42) in which the frequency dependence is neglected, for the two rod gap case. The normalized transfer function \( T(\omega) \) of the system is also plotted. This transfer function is Lorentzian [19] with 3dB half width equal to \( \Delta \omega_{3dB} = 1/r = |K_f|^2 \) and for the case considered, \( \Delta \omega_{3dB}/\omega_0 \approx 3.4 \times 10^{-4} \). Note that the transfer function is displaced by \( \Delta f_0/f_0 = \Delta \omega_0/\omega_0 \approx 5 \times 10^{-4} \) away from the resonant frequency of the mode of the isolated cavity \( f_0 = 193.8\text{THz} \), due to the waveguide-cavity coupling. As observed from the figure, the coupling coefficient \( K_f \) is very well approximated by (42) within the entire 3dB bandwidth of the transfer function.

To ascertain the accuracy of the coupled mode theory, the results for the case of single rod spacing between the cavity and the waveguide obtained by FDTD simulations and the coupled mode theory can be compared. The device under simulation is shown in figure 8. The grid \( \Delta \) for the FDTD is \( \Delta = r_a/10 = 10.8\text{nm} \) and the shaded portions in the figure correspond to the Perfectly
Matched Layers (PMLs) [33]. The fundamental waveguide is excited using the total field/reflected field formulation [15] with a Gaussian pulse having an 1/e intensity of 10fs. As time elapses the values of the electric field are recorded at the waveguide input and output. The input and output waveforms are then numerically Fourier transformed to obtain the transfer function of the system. In figure 8(b) the transfer function obtained using the FDTD method are plotted along with the Lorentzian transfer function obtained from the coupled mode theory. Note that the Lorentzian transfer function is detuned 550GHz to the right of the isolated cavity mode resonant frequency \( f_0 = 193.8 \text{THz} \) in order to incorporate the detuning due to the waveguide-cavity coupling. The value of \( 1/\tau \) calculated using equation (47) is \( 1/\tau = 3.2 \text{THz} \). Given the inherent inaccuracies of both the FDTD and the PWE methods used to calculate the resonant frequency the agreement between the two methods is sufficient to validate the coupled mode method even in the case of the 1 rod gap where the coupling is the strongest.

VIII. MULTI-MODE OR MULTIPLE CAVITIES

The analysis presented above can be applied for the derivation of the evolution equations of the modal amplitudes of a multimode cavity or and multiple cavities side-coupled to a waveguide. Let the field distribution of the \( i^{th} \) mode \( (1 \leq i \leq M) \) of the cavity be \( (E_i, H_i) \) and let \( \omega_i \) be its resonant frequency. The modes obey the Maxwell’s equations \( \nabla \times E_m = j\omega_m \mu H_m \) and \( \nabla \times H_m = -j\omega_m \varepsilon E_m \). If the waveguide is coupled to a series of isolated cavities, then the dielectric constants \( \varepsilon_m \) represent the dielectric constants of the each of the cavities isolated from its surroundings. If the waveguide is coupled to a multimode cavity then \( \varepsilon_m = \varepsilon_c \) where \( \varepsilon_c \) is the dielectric constant distribution of the multimode cavity.

Instead of (2a) and (2b), we use the expansion

\[
E(r; \omega) = \sum_n a_n E_n + a_f E_f + a_b E_b
\]

\[
H(r; \omega) = \sum_n a_n H_n + a_f H_f + a_b H_b
\]

Defining the vector \( a = (a_1, \ldots, a_M) \) to contain the amplitudes of the cavity modes, equations (21) and (23) are written as

\[
\frac{\partial a_f}{\partial z} = j a \cdot k_f, \quad \frac{\partial a_b}{\partial z} = -j a \cdot k_b
\]

(53)

where the vectors \( k_f \) and \( k_b \) are given by \( k_f = (\kappa_{f1}, \ldots, \kappa_{fM}) \) and \( k_b = (\kappa_{b1}, \ldots, \kappa_{bM}) \) where \( \kappa_{fx} \) and \( \kappa_{bx} \) are the coupling coefficients of the \( n^{th} \) cavity mode with the forward and the backward waveguide mode. These coefficients are of the same form as \( \kappa_f \) and \( \kappa_b \) in (22) and (24) except that the fields \( E_r \) and \( H_r \) are replaced by \( E_m \) and \( H_m \), \( \omega_0 \) is replaced by \( \omega_n \) and \( \varepsilon_c \) is replaced by \( \varepsilon_m \). A similar expression to (53) generalizes the coupled mode equation for the backward mode given by (23). Equation (27) is now written in vector notation:
\[ j \int_1^l \, dz_j \mathbf{K}_j^* + j \int_1^l \, dz_a \mathbf{K}_a^* + j \mathbf{K} \cdot \mathbf{a} = 0 \] (54)

where the elements of the coupling matrix \( \mathbf{K} = [\kappa_{mn}] \) are given by

\[ \kappa_{mn} \cong (\omega - \omega_m) \delta_{mn} + \Delta \kappa_{mn} (1 - \delta_{mn}) \] (55)

In (55) \( \delta_{mn} \) is Kronecker’s delta and the coupling coefficients \( \Delta \kappa_{mn} \) are given by

\[ \Delta \kappa_{mn} = \mu (\omega - \omega_m) \int_V \mathbf{H}_m^* \cdot \mathbf{H}_n \, dV + \int_V (\omega \varepsilon_r \varepsilon_m - \omega_m \varepsilon_m) \mathbf{E}_m^* \cdot \mathbf{E}_n \, dV \] (56)

To derive (55) it was assumed that the cavity modes are normalized to have unity energy and that the self-coupling of the cavity modes in the waveguide is negligible as in the case of (44). Applying integration by parts, one can transform (54) to:

\[ (j \mathbf{K} + \mathbf{J}) \cdot \mathbf{a} + \mathbf{K}_f a_f + \mathbf{K}_b b_b = 0 \] (56)

The elements \( j_{mn} \) of the matrix \( \mathbf{J} \) are given by

\[ j_{mn} = j \int_{-l}^l \left( \kappa_{mn} \Lambda_{jmn}^* - \kappa_{nm} \Lambda_{jm}^* \right) \, dz \] (57)

where \( \Lambda_{jmn} \) and \( \Lambda_{jm} \) are the elements of the vector functions \( \mathbf{\Lambda} = [\Lambda_j] dz \) and \( \mathbf{\Lambda} = [\Lambda_j] dz \) defined so that \( \Lambda_j(l) = \Lambda_j(-l) = 0 \). The coupling vectors \( \mathbf{K}_f \) and \( \mathbf{K}_b \) are equal to \( -j \Lambda_j^*(-l) \) and \( j \Lambda_j^*(l) \) respectively. Assuming that the resonant frequencies of the modes are close to each other and that \( \omega \approx \omega_m \), one obtains a system of coupled equations in the time domain

\[ \frac{d\mathbf{a}}{dt} = -j \omega \mathbf{m} \mathbf{a} + \sum_{n} j_{mn} \mathbf{a}_n + j \sum_{n \neq m} \Delta \kappa_{mn} \mathbf{a}_n + \mathbf{K}_{fn} \mathbf{a}_f + \mathbf{K}_{bn} \mathbf{a}_b \] (58)

These equations can describe the evolution of the modal amplitudes in the time domain. The coefficients \( j_{mn} \) constitute a change in the coupling coefficients of the cavity modes induced by the coupling of the cavity modes through the waveguide.

As a first example of a multimode cavity, let us assume a micro-disk/waveguide system as in Figure 2. The modes of the micro-disk can be obtained approximately as in [34] by assuming that the electric field modes \( \mathbf{E} = y \mathbf{E}_y \) are given by \( \mathbf{E}_y = J_{q} \frac{e^{jR \rho}}{R} \) inside the microdisk, where \( R \) is the disk radius, \( k_1 = 2 \pi n \varepsilon / \lambda \), \( n \varepsilon \) is the effective index of the core region of the slab in which the system is buried, \( \rho = (x^2 + z^2)^{1/2} \), \( q = \tan^{-1}(x/z) \) and \( \beta \) is the propagation constant in the azimuthal direction with \( \beta R = q \) where \( q \) is an integer. Given the \( e^{j\omega t} \) time dependence the \( e^{jR \rho} \) variation corresponds to the mode clockwise propagating mode inside the micro-disk, while the \( e^{jR \rho} \) corresponds to the counterclockwise propagating mode. These two modes are degenerate and have the same resonant frequency \( \omega_0 \). Outside the disc the mode is assumed to decay exponentially [34] as \( \exp(-\alpha(\rho-R)) \),
where the damping constant is \( a = (\beta^2 - k_z^2)^{1/2} \) and \( k_z = 2\pi n_2 \), \( n_2 \) being the refractive index outside the micro-disk. Matching the tangential electric and magnetic components, one obtains the approximate characteristic equation for the micro-disk:

\[
k_z J_{q+1}(k_z R) = \left( a + \frac{q}{R} \right) J_q(k_z R)
\]

Solving (59) numerically one can obtain an approximate value for the resonant frequency \( \omega_0 \) of the modes. The modes of the waveguide can be approximately obtained using the effective index method from the modes of the slab whose core refractive index is \( n_{\text{effv}} \), its cladding index is \( n_2 \) and whose thickness equals the waveguide width.

As defined, the modes of the resonator are no longer real and hence equation (47) no longer applies. Although one could select the waveguide modes to be real by using \( \cos(\beta R\phi) \) and \( \sin(\beta R\phi) \) instead of \( e^{\pm j\beta R\phi} \), the use of traveling wave solution simplifies the situation as will be shown next. Figure 9 depicts the values of the functions \( \Lambda_{R_1}(z) \), for the counterclockwise \((n=1)\) and the clockwise \((n=2)\) propagating wave, for \( q=20 \). The disc radius is \( R=0.3\mu m \), the vertical effective index is taken \( n_{\text{effv}}=2.74 \), the waveguide width is \( w=0.3\mu m \) and the gap between the waveguide and the micro-disk is \( g=0.2\mu m \), while the cladding index is \( n_2=1 \). The values of the functions \( \Lambda_{R_0}(z) \) are also plotted. Since \( K_{i2} = -j\Lambda_{R_2}^\ast(l) \approx 0 \), it is deduced that the coupling between the forward waveguide mode and the counterclockwise propagating mode is negligible. Also, since \( K_{i1} = j\Lambda_{R_1}^\ast(l) \approx 0 \), the same is true for the clockwise propagating waveguide mode and the backward propagating waveguide mode. The \( J \) matrix in this case turns out to be diagonal with \( j_{11} = j_{22} = l = -(2.09+j7.47) \times 10^{11} \text{Hz} \). Also \( \Delta\kappa_{12} \approx \Delta\kappa_{21} \approx 0 \) since the coupling of the resonator modes inside the waveguide is negligible. The system of coupled equations in the time domain is therefore written as

\[
\frac{d\vec{a}_i}{dt} = -j\omega_0\vec{a}_i + K_{ij}\vec{a}_j + l\vec{a}_l \quad (60)
\]

\[
\frac{d\vec{a}_j}{dt} = -j\omega_0\vec{a}_j + K_{k2}\vec{a}_k + l\vec{a}_l \quad (61)
\]

Hence the detuning for both resonator modes can be computed numerically and is found \( \Delta f = \Delta\omega_0/(2\pi) = -\text{Im}\{|l|/(2\pi) \approx 118\text{GHz} \) while the resonant frequency \( f_0 \) of the modes of the isolated micro-disk resonator is \( f_0 = 199.7\text{THz} \) for \( q=20 \). Using the fact that \( \text{Re}\{|l| \approx |K_{ji}|^2/2 = 1/\tau \), one can easily show that the amplitude transfer function is given by

\[
H(\omega) = \frac{a_j(l)}{a_{f_0}} = \frac{j\tau(\omega - \omega_1) - 1}{j\tau(\omega - \omega_1) + 1} \quad (62)
\]
with \( \omega_1 = \omega_0 + \Delta \omega_0 \). Note that the power transfer function \( T(\omega) = |H(\omega)|^2 = 1 \), and hence the spectrum of the signal simply undergoes a phase change near \( \omega_1 \). Since the backward waveguide mode is not excited, no power is reflected back and to the device input.

The portion of the spectrum near \( \omega_1 \) excites the resonator modes whose energy is gradually transferred to the forward waveguide propagating mode.

As a second example, consider a photonic crystal based SCISSOR consisting of four defect cavities such as the one depicted in figure 10. In this case we expand the field in terms of the modes of the isolated cavities, i.e. \( E_m = E_0(r-nz; \mathbf{z}), H_m = H_0(r-nz; \mathbf{z}) \), \( \omega_m = \omega_0 \) and \( \varepsilon_m = \varepsilon_c \) (\( r-nr_1 \)), where \( E_0, H_0, \omega_0 \) and \( \varepsilon_c \) is the electric field mode, the magnetic field mode, the resonant frequency and the dielectric constant of an isolated cavity respectively. One obtains

\[
\kappa_f(z) \cong e^{-j\beta z} \kappa_f(z - nz_1)
\]  

(63)

with \( \kappa_f \) given by (22), by replacing \( E_r \) and \( H_r \) by \( E_0 \) and \( H_0 \) respectively. A similar expression hold for \( \kappa_{bn} \) and the vector \( K_f, K_b \) as well as the matrices \( K \) and \( J \) can be computed numerically. Note that (63) can be used to obtain the coupling coefficients \( \kappa_{fn} \) from the coupling coefficients of a single cavity/waveguide system without additional computational cost. Using (53) one can obtain the transfer function for the reflected wave at the waveguide input as

\[
\frac{a_l(I)}{a_{j0}} = H_r = -K^*_b \cdot \left\{ (jK + J)^{-1} \cdot K_f \right\}
\]  

(64)

Figure 10(b) depicts the power transfer function for the reflected wave for the SCISSOR of figure 10(a). The four coupled cavities of the SCISSOR can be viewed as a large cavity, as illustrated in Fig. 10(a). This larger cavity supports four modes whose resonant frequencies slightly differs from \( \omega_0 \) and each mode produces a resonance in the reflected spectrum. This explains the four resonances in Fig. 10(b).

IX. CONCLUSIONS

In this paper, a systematic derivation of the coupled mode evolution equations was presented starting from Maxwell’s equations and using reciprocity relations for both constant cross-section and periodically varying waveguides as well. This allowed the derivation of the coupling coefficients, including their frequency dependence. It was shown that the presence of the waveguide causes a detuning in the cavity resonant frequency. This detuning was estimated in the case of a photonic crystal cavity coupled to a photonic crystal waveguide and it was shown that it can be significant, especially if the structure is to be used for filtering applications. Finally the coupled mode evolution equation was derived for the case of multimode or multiple cavities as well.
APPENDIX A: RECIPROCITY RELATIONS

In this appendix the reciprocity relations used in the derivation of (1) are briefly summarized. Assuming two electromagnetic fields \((\mathbf{E}_1, \mathbf{H}_1)\) and \((\mathbf{E}_2, \mathbf{H}_2)\) that obey Maxwell’s equations:

\[
\nabla \times \mathbf{E}_1 = j \omega_1 \mu \mathbf{H}_1 \tag{A1}
\]
\[
\nabla \times \mathbf{H}_1 = -j \omega_1 \varepsilon \mathbf{E}_1 \tag{A2}
\]
\[
\nabla \times \mathbf{E}_2 = j \omega_2 \mu \mathbf{H}_2 \tag{A3}
\]
\[
\nabla \times \mathbf{H}_2 = -j \omega_2 \varepsilon \mathbf{E}_2 \tag{A4}
\]

Then defining the vector function

\[
\mathbf{F}_{12} = \mathbf{E}_1 \times \mathbf{H}_2^* + \mathbf{E}_2^* \times \mathbf{H}_1 \tag{A6}
\]

one can show that

\[
\oint_{\partial V} \mathbf{F}_{12} \cdot d\mathbf{s} = j \mu (\omega_1 - \omega_2) \int_{V} \mathbf{H}_2^* \cdot \mathbf{H}_1 \, dV + j \int_{V} (\varepsilon_1 \omega_1 - \varepsilon_2 \omega_2) \mathbf{E}_2^* \cdot \mathbf{E}_1 \, dV \tag{A5}
\]

where \(V\) is a volume enclosed by the closed surface \(\partial V\). It can also be shown that

\[
\frac{\partial}{\partial z} \oint_{\partial S} \mathbf{F}_{12} \cdot d\mathbf{l} = j \mu (\omega_1 - \omega_2) \int_{S} \mathbf{H}_2^* \cdot \mathbf{H}_1 \, dV + j \int_{S} (\varepsilon_1 \omega_1 - \varepsilon_2 \omega_2) \mathbf{E}_2^* \cdot \mathbf{E}_1 \, dV \tag{A6}
\]

where \(S\) is a surface enclosed by the closed contour \(\partial S\) and \(d\mathbf{l}\) is tangential to \(\partial S\).

APPENDIX B: VOLUME INTEGRALS OF ELECTROMAGNETIC FIELDS

Assuming two electromagnetic fields obeying (A1)-(A4). By defining the vector function

\[
\mathbf{G}_{12} = \mathbf{E}_1 \times \mathbf{H}_2^* \tag{B1}
\]

If either of the fields vanishes on \(\partial V\) then \(\int_{V} \nabla \cdot \mathbf{G}_{12} = 0\) and consequently

\[
\mu \int_{V} \mathbf{H}_2^* \cdot \mathbf{H}_1 \, dV = \frac{\omega_2}{\omega_1} \int_{V} \varepsilon_2 \mathbf{E}_2^* \cdot \mathbf{E}_1 \, dV \tag{B2}
\]

APPENDIX C: ORTHOGONALITY RELATIONS OF CAVITY MODES

The electromagnetic mode \((\mathbf{E}_i, \mathbf{H}_i)\) of a multimode cavity obey the following Maxwell’s equations:
\[ \nabla \times \mathbf{E}_i = j \omega_i \mu \mathbf{H}_i \quad (C1) \]
\[ \nabla \times \mathbf{H}_j = -j \omega_j \varepsilon \mathbf{E}_j \quad (C2) \]

where \( \omega_i \) is the resonant frequency of the mode. Defining the vector functions \( \mathbf{F}_{ik} = \mathbf{E}_i \times \mathbf{H}_k^* + \mathbf{E}_k^* \times \mathbf{H}_i \) and \( \mathbf{G}_{ik} = \mathbf{E}_i \times \mathbf{H}_k^* - \mathbf{E}_j^* \times \mathbf{H}_k \) and setting \( \int_V \nabla \cdot \mathbf{G}_{ik} \, dV = 0 \) and \( \int_V \nabla \cdot \mathbf{F}_{ik} \, dV = 0 \) one can show that the modes obey the following orthogonality relations

\[ \mu \int_V \mathbf{H}_j^* \cdot \mathbf{H}_k \, dV = \int_V \varepsilon \mathbf{E}_i^* \cdot \mathbf{E}_j \, dV = 0 \text{ if } i \neq k \quad (C3) \]

**VALUES OF RESONANT FREQUENCY DETUNING FOR VARIOUS WAVEGUIDE-CAVITY SPACINGS**

<table>
<thead>
<tr>
<th>Spacing</th>
<th>( \Delta f_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 rods</td>
<td>547GHz   0.2%</td>
</tr>
<tr>
<td>2 rods</td>
<td>102GHz   0.05%</td>
</tr>
<tr>
<td>3 rods</td>
<td>22GHz    0.01%</td>
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</table>

**REFERENCES**


**Figure Captions**

Figure 1: a) a Coupled Resonator Optical Waveguides (CROW) coupled to two photonic crystal waveguides, and b) a Side coupled integrated sequence of resonators (SCISSOR).

Figure 2: Geometry of the waveguide-cavity problem.

Figure 3: Dielectric constant distributions involved in the calculation of the coupling coefficients $\kappa_f$ and $\kappa_b$: a) the cavity/waveguide system distribution $\varepsilon$, b) the cavity distribution $\varepsilon_c$, c) the waveguide distribution $\varepsilon_w$, d) the difference $\varepsilon - \varepsilon_c$. The overlap integrals are calculated on the shaded regions of (d) which are the only non-zero regions of $\varepsilon - \varepsilon_c$.

Figure 4: a) Isolated cavity mode distribution $E_r = E_y$ and b) isolated waveguide mode distribution $|E_f|$ both depicted on the waveguide-cavity system.

Figure 5: Values of the coupling coefficient $\kappa_f$ calculated using (22) and numerical integration.

Figure 6: Values of the coupling coefficient $\Lambda_f$ calculated using (22) and numerical integration.

Figure 7: Values of the coupling coefficient $K_f$ calculated using either equation (42) or (40).

Figure 8: a) Geometry used in the FDTD simulations, b) Comparison of the transfer functions obtained by the FDTD and the Coupled Mode Theory methods.

Figure 9: The functions $\Lambda_{fn}$ and $\Lambda_{bn}$ for the micro-disk/waveguide system described in the text.

Figure 10: a) A finite SCISSOR consisting of a photonic crystal waveguide side-coupled to four defect cavities, b) The transfer function of the SCISSOR of (a) calculated using (64).
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
Figure 6
$K_f$ value using (42)

$K_f$ value using (40)

Transfer Function

$|T(\omega)|$

Figure 7
Figure 8
Figure 9
Figure 10