Performance Analysis of Differential Phase Shift Keying Optical Receivers in the Presence of In-band Crosstalk Noise

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Abstract—In-band crosstalk can pose important limitations in an all-optical Wavelength Division Multiplexed (WDM) network. Recent studies have demonstrated that Differential Phase Shift Keying (DPSK), can tolerate higher in-band crosstalk noise levels compared to Amplitude Shift Keying (ASK). In this paper, the performance of a DPSK receiver, limited by in-band crosstalk noise is studied theoretically. The model takes into account both the in-band crosstalk noise as well as the Amplified Spontaneous Emission (ASE) noise of the optical amplifiers. The model is based on the evaluation of the Moment Generating Function (MGF) of the decision variable through which, the Error Probability (EP) can be calculated by applying the saddle-point approximation. This provides a rigorous model for the evaluation of the EP of a DPSK receiver in the presence of ASE and in-band crosstalk noises. In the absence of the ASE noise, a closed form formula for the EP is also given which is useful for estimating the error floor set by the in-band crosstalk noise.

Index Terms—Wavelength division multiplexing, crosstalk, error analysis, optical receivers.
I. INTRODUCTION

The performance of Wavelength Division Multiplexing (WDM) networks can be degraded by the presence of in-band crosstalk noise [1]. This noise arises at optical cross-connects because, due to their imperfect filtering characteristics, a small delayed version of the signal or a small portion of light from other channels at the same frequency (in a network with wavelength reuse) is routed along the same path as the signal. Since in-band crosstalk noise is at the same wavelength as the signal, it cannot be removed using additional filtering and can degrade the Error Probability (EP) at the receiver. The power of the noise is proportional to that of the signal and hence an increase of the signal power, does not change the signal-to-crosstalk ratio (defined as the ratio of the power of the signal to the power of the crosstalk). This sets a lower limit in the value of the EP [2], usually referred to as an error floor, which can limit the number of nodes of a WDM network.

Due to its importance, in-band crosstalk noise has been extensively studied in the literature (see [3]-[10] and references therein) in the case of Amplitude Shift Keying (ASK) signal modulation. At the receiving photodiode, two in-band noise contributions must be taken into account [7]: one resulting from the beating of the signal with the optical crosstalk noise and one from the beating of the crosstalk noise with itself. The impact of in-band crosstalk was investigated using the Gaussian approximation [3] in the case of an Arrayed Waveguide Grating (AWG) interconnection. The Gaussian approximation is based on the Central Limit Theorem (CLT) and assumes a large number $M$ of interfering crosstalk components. Although the Gaussian approximation is relatively straightforward, it neglects the crosstalk-crosstalk beating noise at the receiving photodiode. An accurate noise description
should consider both in-band crosstalk noise contributions, which are statistically correlated since they originate from the same optical noise. Besides, as shown experimentally by Jiang et al [9], in the presence of in-band crosstalk noise, the Probability Density Function (PDF) of the decision variable at the receiver is asymmetric, and hence the Gaussian model may not provide an accurate description for the noise statistics. It was also shown [10] that, due to the presence of the crosstalk-crosstalk beating, the decision variable asymptotically becomes a Chi-square random variable as \( M \to \infty \). Hence, the in-band crosstalk noise cannot be assumed Gaussian, even in this limit.

Recently, it was experimentally demonstrated that Differential Phase Shift Keying (DPSK) [11] can increase the system tolerance to in-band crosstalk compared to ASK. The above study considered a single interfering component \( (M=1) \). There are however, many practical situations, such as the passive AWG interconnection, where the number of crosstalk components can be quite large. Therefore the performance of the DPSK receiver in the presence of many crosstalk components must be considered. It should also be noted that the DPSK modulation format increases the system tolerance to fiber-induced non-linear distortion [12]. Therefore, the DPSK format deserves a more detailed analysis since it constitutes an attractive candidate for future WDM network implementation.

In this paper, the performance of a DPSK receiver, limited by in-band crosstalk and ASE noises is theoretically analyzed assuming a large number of in-band interfering components. A closed form formula for the Moment Generating Function (MGF) of the decision variable is derived. This formula can be used to estimate the EP using the saddle-point approximation [13]. In addition, in the case where the ASE noise is neglected, a closed form formula is given for the EP. This
formula can be used to estimate the error floor set by the in-band crosstalk noise in a DPSK receiver.

II. RECEIVER MODEL

In this section, the DPSK receiver model in the presence of in-band crosstalk noise will be presented. This model will be used in the next sections to evaluate the decision variable and its MGF. A typical DPSK receiver [13] is depicted in Figure 1. At the output of the optical amplifier, the optical field $S(t)$ is given by

$$ S(t) = S_0(t) + S_X(t) + N_{ase}(t) $$

where $S_0(t)$ is the desired DPSK signal. Assuming two consecutive bit intervals, $S_0(t)$ is written as:

$$ S_0(t) = c_0 \left( e^{i\theta_0 a} g(t) + e^{i\theta_0 b} g(t+T) \right) e^{i\phi_0} $$

for values of $t$ inside the interval $[-T, T]$, where $T$ equals a single bit duration. In (2), the random phase $\phi_0$ is due to the LASER phase noise and is assumed uniform in $[0,2\pi]$, $g(t)$ is an arbitrary pulse shape which is considered zero outside the interval $[0,T]$. If the bit $B_a$ to be transmitted in $[0,T]$ is the same with the bit $B_b$ of the previous interval $[-T,0]$, then $\theta_{0a}-\theta_{0b}=0$, while if $B_b \neq B_a$ then $\theta_{0a}-\theta_{0b}=\pi$, i.e. the information is coded as phase shifts between the successive bit intervals. The constant $c_0$ is the amplitude of the signal. The field $S_X(t)$ is the in-band crosstalk noise, and is written as:

$$ S_X(t) = \sum_{m=1}^{M} c_m \left( e^{i\theta_{ma}} g(t) + e^{i\theta_{mb}} g(t+T) \right) e^{i\phi_m} $$

where $M$ is the total number of crosstalk components, the phases $\theta_{ma}$ and $\theta_{mb}$ designate the bit changes of the crosstalk component $m$, $c_m$ is its amplitude, and $\phi_m$ is due to the phase noise. The phases $\phi_m$ for $1 \leq m \leq M$, are uniform random variables inside $[0,2\pi]$, independent of each other and of $\phi_0$. The field $N_{ase}(t)$ is the ASE noise.
of the optical amplifier and can be modeled as an additive white Gaussian stochastic noise process. The field \( S(t) \) is filtered by the optical filter and the filtered optical field \( S_H(t) \) is given by

\[
S_H(t) = \sum_{m=0}^{M} c_m \left( e^{j\beta_m} g_H(t) + e^{j\beta_m} g_H(t + T) \right) e^{j\phi_m} + n(t)
\]

where \( g_H(t) \) and \( n(t) \) are the filtered versions of \( g(t) \) and \( N_{ase}(t) \) respectively. The field \( S_H(t) \) is directed to a Mach-Zehnder interferometer, which forms two auxiliary optical signals \( S_1(t) = [S_H(t)+S_H(t-T)]/2 \) and \( S_0(t) = [S_H(t) - S_H(t-T)]/2 \). Each signal is fed to a separate photodiode. If the quantum efficiency is taken equal to unity, and if the intensity \( |S_1(t)|^2 \) is measured in photons/s, then the induced photocurrent \( i_1(t) \) at the upper photodiode is \( i_1(t) = \frac{1}{2}|S_1(t)|^2 \) (photoelectrons/s) where the factor \( \frac{1}{2} \) is due to the complex notation adopted for the optical fields [13]. The photocurrent \( i_1(t) \) is directed to an electrical filter. Assuming an integrate-and-dump filter, the photocurrent at the output of the filter at time \( t = T \), is given by

\[
i_{H1}(T) = \frac{1}{T} \int_0^T i_1(t') dt' = \frac{1}{8T} \int_0^T |S_H(t') + S_H(t' - T)|^2 dt
\]

The photocurrent \( i_0(t) = \frac{1}{2}|S_0(t)|^2 \) at the lower photodiode passes through a similar electrical filter and the filtered photocurrent is

\[
i_{H0}(T) = \frac{1}{T} \int_0^T i_0(t') dt' = \frac{1}{8T} \int_0^T |S_H(t') - S_H(t' - T)|^2 dt
\]

The filtered photocurrents \( i_{H1}(t) \) and \( i_{H0}(t) \) are then fed to a decision circuit. At \( t = T \), the value of the decision variable \( D = D(T) = i_{H1}(T) - i_{H2}(T) \) is used to infer the value of the received bit. Using (5) and (6) it is easy to show that

\[
D = \frac{1}{2T} \text{Re} \left\{ \int_0^T S_H(t') S_H^*(t' - T) dt' \right\}
\]
Equations (1)-(7) describe the DPSK receiver model and will be used to derive a closed form expression for the Moment Generating Function (MGF) of the decision variable \( D, M_D(s)=\langle e^{iD} \rangle \) in the presence of the ASE noise.

III. ESTIMATION OF THE DECISION VARIABLE

Using (7), one can derive an expression for the decision variable that contains both the ASE and the in-band crosstalk contribution. To accomplish this, the ASE noise is first expanded in a Fourier series. Then, four auxiliary random variables are defined which contain the contribution of the crosstalk noise, and the decision variable is expressed in terms of these random variables.

A. Expansion of the ASE Noise

To estimate the decision variable in the presence of the ASE noise, the amplifier noise after the optical filter \( n_1(t)=n(t) \) inside \([0,T]\) is written as [13]

\[
n_1(t) = \sum_{k=-\infty}^{\infty} N_k e^{j2\pi kt/T} \tag{8}
\]

where \( N_k=N_{kr}+jN_{ki} \) are the Fourier components of \( n_1(t) \) and \( N_{kr} \) and \( N_{ki} \) are independent Gaussian random variables [13] having zero mean value and standard deviation

\[
\langle N_{kr}^2 \rangle = \langle N_{ki}^2 \rangle = \sigma_k^2 = n_{sp}(G-1)H^2 (k/T)/T \tag{9}
\]

where \( H(k/T) \) is the transfer function of the optical filter at \( f=k/T \), which for simplicity is assumed real. The ASE noise inside \([-T,0]\) \( n_2(t) \) can be written in a similar manner in terms of its Fourier components \( M_k=M_{kr}+jM_{ki} \).

\[
n_2(t) = \sum_{k=-\infty}^{\infty} M_k e^{j2\pi kt/T} \tag{10}
\]

The standard deviation of \( M_{kr} \) and \( M_{ki} \) is the same as that of \( N_{kr} \) and \( N_{ki} \). The pulse \( g_{1A}(t) \) can also be written as
\[
g_{\mathcal{H}}(t) = \sum_{k=-\infty}^{+\infty} g_k e^{\frac{j2\pi kt}{T}}
\]  

(11)

where \( g_k \) are the Fourier coefficients of \( g_{\mathcal{H}}(t) \).

**B. Definition of the Auxiliary Random Variables**

To facilitate the analysis, four auxiliary random variables are defined:

\[
X_a = c_0 e_{0a} \cos \phi_0 + \sum_{m=1}^{+\infty} c_m e_{ma} \cos \phi_m
\]

(12a)

\[
X_b = c_0 e_{0b} \cos \phi_0 + \sum_{m=1}^{+\infty} c_m e_{mb} \cos \phi_m
\]

(12b)

\[
Y_a = c_0 e_{0a} \sin \phi_0 + \sum_{m=1}^{+\infty} c_m e_{ma} \sin \phi_m
\]

(12c)

\[
Y_b = c_0 e_{0b} \sin \phi_0 + \sum_{m=1}^{+\infty} c_m e_{mb} \sin \phi_m
\]

(12d)

where \( e_{ma} = \exp(j\theta_{ma}) \) and \( e_{mb} = \exp(-j\theta_{mb}) \). The field \( S_{\mathcal{H}}(t) \) inside \([0,T]\) is

\[
S_{\mathcal{H}}(t) = \sum_k S_{ka} e^{\frac{j2\pi kt}{T}} = \sum_k \left[ (X_a + jY_a) g_k + N_k \right] e^{\frac{j2\pi kt}{T}}
\]

(13)

where \( t \in [0,T] \) and \( S_{ka}=(X_a+jY_a)g_k+N_k \). To derive (13), one uses (8), (11) and (12).

Similarly, the optical field \( S_{\mathcal{H}}(t-T) \) is written as

\[
S_{\mathcal{H}}(t-T) = \sum_k S_{kb} e^{\frac{j2\pi kt}{T}} = \sum_k \left[ (X_b + jY_b) g_k + M_k \right] e^{\frac{j2\pi (t-T)k}{T}}
\]

(14)

where \( S_{kb}=(X_b+jY_b)+M_k \). Equation (13) will serve to deduce one fundamental difference between the ASE and the in-band crosstalk noise. The ASE noise before the optical filter, is an additive white Gaussian noise and hence the real and imaginary part of its spectral components \( N_k, M_k \) are uncorrelated. On the contrary, the in-band crosstalk noise is not white and the spectral components \( S_{ka} \) and \( S_{kb} \) are correlated in general. Hence, when both noises are present, the total noise can behave quite differently than a white Gaussian process and the statistical behavior of the decision
variable can be different as discussed in section IV.C.

One important difference between the ASE noise and the crosstalk noise is that the crosstalk noise can be much more bursty (see Ref. [11]). This may cause system outages in systems using forward-error-correction (FEC). So, the cross-talk induced error floor is quite important and may be related to system outage.

C. Derivation of the decision variable

Applying (13) and (14) and integrating, it is easy to show that

\[
\frac{1}{T} \int_{0}^{T} S_H(t')S_H(t'-T)dt' = \sum_{k} \left\{ [(\alpha + j\beta_{a}) g_k + N_{k} \left\{ (\alpha - j\beta_{b}) g_k^* + M_{k} \right\}] \right\}
\]

Taking the real part of (15) and using (7) one obtains,

\[
D = \frac{\gamma}{2} \sum_{k} \left\{ (x_{ka} + N_{ka}) (x_{kb} + M_{kb}) + (y_{ka} + N_{ka}) (y_{kb} - M_{kb}) \right\}
\]

where

\[
x_{ka} = \text{Re} \{(X_{a} + jY_{a}) g_k \} \tag{17a}
\]

\[
x_{kb} = \text{Re} \{(X_{b} + jY_{b}) g_k \} \tag{17b}
\]

\[
y_{ka} = \text{Im} \{(X_{a} + jY_{a}) g_k \} \tag{17c}
\]

\[
y_{kb} = \text{Im} \{(X_{b} + jY_{b}) g_k \} \tag{17d}
\]

Defining

\[
D_{kr} = \frac{1}{2} (x_{ka} + N_{kr}) (x_{kb} + M_{kr}) \tag{18a}
\]

\[
D_{ki} = \frac{1}{2} (y_{ka} + N_{ki}) (y_{kb} - M_{ki}) \tag{18b}
\]

the decision variable is written as

\[
D = \sum_{k} (D_{kr} + D_{ki}) \tag{19}
\]
Equations (18)-(19) indicate the fact, that the decision variable is the sum of the random variables $D_{kr}$ and $D_{ki}$. Given the values of $x_{ka}, x_{kb}, y_{ka}$ and $y_{kb}$, it is deduced that $D_{ki}$ and $D_{kr}$ are mutually independent and each one is a product of two independent Gaussian random variables. However, $x_{ka}, x_{kb}, y_{ka}$ and $y_{kb}$ are also random variables whose statistical behavior is determined by the statistics of $X_a, Y_a, X_b$ and $Y_b$.

IV. EVALUATION OF THE MGF OF THE DECISION VARIABLE

Equation (19) is the starting point for the evaluation of the MGF, $M_D(s)=\langle e^{sD}\rangle$, of the decision variable $D$. To evaluate the MGF, first the expectation of $e^{sD}$ with respect to the ASE noise components, $M_k$ and $N_k$ will be calculated. Then, the statistical properties of $X_a, Y_a, X_b$ and $Y_b$ will be used to evaluate the expectation of $e^{sD}$ with respect to these variables as well.

A. Expectation of $e^{sD}$ with respect to the ASE components

The MGFs $M_{kr}(s)$ and $M_{ki}(s)$ of $D_{kr}$ and $D_{ki}$ respectively, can be computed in closed form using the formula

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{a(x^2+y^2)+\beta xy+\delta y+\epsilon}$$

$$= \frac{1}{\sqrt{4a^2-\beta^2}} \exp \left( \frac{\beta\gamma\delta + a(\gamma^2 + \delta^2)}{4a^2 - \beta^2} + \epsilon \right)$$

where $a<0$. This formula is easily derived using [19, eq. 3.323.2]. Using (20), one can show that,

$$M_{kr}(s) = \frac{1}{\sqrt{1-\sigma^2_{kr}s^2/4}} \exp \left( \frac{(\gamma_{ka}^2 + \gamma_{kb}^2 \sigma_{kr}^2 s^2/4 + x_{ka}x_{kb}s)}{2(1-\sigma^2_{kr}s^2/4)} \right)$$

$$M_{ki}(s) = \frac{1}{\sqrt{1-\sigma^2_{ki}s^2/4}} \exp \left( \frac{(\gamma_{ka}^2 + \gamma_{kb}^2 \sigma_{ki}^2 s^2/4 + y_{ka}y_{kb}s)}{2(1-\sigma^2_{ki}s^2/4)} \right)$$
The MGF $M_{DC}(s)$ of $D$ given the realization of the crosstalk noise, i.e. the values of $x_{ka}, y_{ka}, x_{kb}, y_{kb}$, is simply the product of these MGFs, i.e.

$$M_{DC}(s) = \prod_{k=-\infty}^{\infty} M_{kr}(s) \prod_{k=-\infty}^{\infty} M_{kr}(s)$$ (22)

Substituting (20) and (21) in (22), one obtains

$$M_{DC}(s) = \left(\prod_{k=-\infty}^{\infty} \frac{1}{1-\sigma_k^2 s^2/4}\right) \times \exp\left(\sum_{k=-\infty}^{\infty} \left(\frac{x_{ka}^2 + x_{kb}^2 + y_{ka}^2 + y_{kb}^2}{2(1-\sigma_k^2 s^2/4)} \right) s \right)$$ (23)

Substituting (17) into (23), the following form for $M_{DC}(s)$ is obtained

$$M_{DC}(s) = C(s) \exp\left(A(s) \left[X_a^2 + Y_a^2 + X_b^2 + Y_b^2\right] + B(s) \left[X_a Y_b + Y_a X_b\right]\right)$$ (24)

where

$$A(s) = \sum_{k=-\infty}^{\infty} \frac{\sigma_k^2 s^2 |g_k|^2}{8(1-\sigma_k^2 s^2/4)}$$ (25a)

$$B(s) = \sum_{k=-\infty}^{\infty} \frac{|g_k|^2 s}{2(1-\sigma_k^2 s^2/4)}$$ (25b)

$$C(s) = \prod_{k=-\infty}^{\infty} \frac{1}{1-\sigma_k^2 s^2/4}$$ (25c)

To complete the evaluation of the MGF the expected value of (24) with respect to $X_a, Y_a, X_b$ and $Y_b$ must be computed and thus, the statistical properties of the auxiliary random variables $X_a, Y_a, X_b$ and $Y_b$ must first be considered.

**B. Statistical Behavior of the Auxiliary Random Variables**

Given the value of $\phi_0, e_{0a}, e_{0b}$, the mean values of the auxiliary variables are given by
\[ m_{xa} = \langle X_a \rangle = c_0 e_{0a} \cos \phi_0 \]  
\[ m_{xb} = \langle X_b \rangle = c_0 e_{0b} \cos \phi_0 \]  
\[ m_{ya} = \langle Y_a \rangle = c_0 e_{0a} \sin \phi_0 \]  
\[ m_{yb} = \langle Y_b \rangle = c_0 e_{0b} \sin \phi_0 \]  

Equations (26a)-(26d) are easily derived, since \( \langle \cos \phi_m \rangle = \langle \sin \phi_m \rangle = 0 \), for \( m > 0 \). To evaluate the standard deviations of \( \sigma_{xa}, \sigma_{xb}, \sigma_{ya}, \sigma_{yb} \), of the auxiliary random variables \( X_a, X_b, Y_a \), and \( Y_b \) respectively, one uses \( \langle \cos^2 \phi_m \rangle = \langle \sin^2 \phi_m \rangle = \frac{1}{2} \) and \( \langle e_{0a}^2 \rangle = \langle e_{0b}^2 \rangle = 1 \). The result is

\[ \sigma_{xa}^2 = \sigma_{ya}^2 = \sigma_{xb}^2 = \sigma_{yb}^2 = \frac{1}{2} \sum_{m \geq 1} c_m^2 \]  

As \( M \to \infty \), the random variables \( X_a, X_b, Y_a, Y_b \) asymptotically become Gaussian as a consequence of the Central Limit Theorem (CLT). It is interesting to also note that as \( M \to \infty \), the auxiliary random variables become statistically independent of each other. To show this, the covariance \( \rho_{kl} = \langle (Z_k - \langle Z_k \rangle)(Z_l - \langle Z_l \rangle) \rangle \) where \( Z_1 = X_a, Z_2 = X_b, Z_3 = Y_a \) and \( Z_4 = Y_b \) can be used. If the covariance \( \rho_{kl} \) is zero for \( k \neq l \), then (due to the Gaussian statistics of the variables) the variables are independent [18]. The evaluation of \( \rho_{kl} \) is relatively straightforward. For example for \( k=1 \) and \( l=2 \), one has

\[ \rho_{12} = \langle (X_a - m_{xa})(X_b - m_{xb}) \rangle = \sum_{m=1}^{M} \sum_{n=1}^{M} c_m c_n \langle \cos \phi_m \cos \phi_n \rangle \langle e_{ma} e_{nb} \rangle \]  

The variables \( \phi_m \) are independent and since \( \langle \cos \phi_m \rangle = 0, \langle \cos^2 \phi_m \rangle = \frac{1}{2} \), one obtains \( \langle \cos \phi_m \cos \phi_n \rangle = \frac{1}{2} \delta_{mn} \) where \( \delta_{mn} \) is Kronecker’s delta (\( \delta_{mn} = 0 \) for \( m \neq n \) and \( \delta_{nn} = 1 \)). Using the fact that \( \langle e_{ma} e_{mb} \rangle = \langle \exp(i(\theta_{ma} - \theta_{mb})) \rangle = \frac{1}{2}(+1) + \frac{1}{2}(-1) = 0 \), it is easy to ascertain that \( \rho_{12} = 0 \). In a similar manner, it is possible to show that \( \rho_{kl} = 0 \) for \( k \neq l \). This implies that the auxiliary Gaussian variables \( X_a, X_b, Y_a, \) and \( Y_b \) are independent.
C. Estimation of the MGF of the decision variable

To complete the estimation of the MGF, the expected value of (24) with respect to \(X_a, X_b, Y_a,\) and \(Y_b\) must be considered. Applying the independence of \(X_a, X_b, Y_a,\) and \(Y_b,\) the expected value of (24) becomes

\[
M_D(s) = \langle M_{D|c}(s) \rangle = C(s)M_X(s)M_Y(s) \tag{29}
\]

where \(M_A(s)\) and \(M_Y(s)\) are given by

\[
M_X(s) = \langle \exp(A(s)[X_a^2 + X_b^2] + B(s)X_aX_b) \rangle \tag{30a}
\]

\[
M_Y(s) = \langle \exp(A(s)[Y_a^2 + Y_b^2] + B(s)Y_aY_b) \rangle \tag{30b}
\]

The expectations (30a)-(30b) involve the calculation of 2D integrals of a function of the form \(e^{p(x,y)}\) where \(p(x,y)\) is a second order polynomial with respect to \(x\) and \(y.\)

These integrals can be computed using (20), and the result for \(M_D(s)\) is

\[
M_D(s) = M_0(s) = \frac{C(s)}{\left(1 - 2A(s)s^2 \right)^2 - \left( B(s)s^2 \right)^2}
\times \exp\left( \frac{c_0^2 \left[ 2A(s) + B(s) \right]}{1 - (2A(s) + B(s))s^2} \right) \tag{31}
\]

in the case where \(\theta_{ma} = \theta_{mb} = 0.\)

\[
M_D(s) = M_\pi(s) = \frac{C(s)}{\left(1 - 2A(s)s^2 \right)^2 - \left( B(s)s^2 \right)^2}
\times \exp\left( \frac{c_0^2 \left[ 2A(s) - B(s) \right]}{1 - (2A(s) - B(s))s^2} \right) \tag{32}
\]

in the case where \(\theta_{ma} - \theta_{mb} = \pi.\) Equations (31) and (32) are the required MGFs of the decision variable of a DPSK receiver, taking into account both the ASE and the in-band crosstalk noises.
Note that since $A(s)=A(-s)$, $B(s)=-B(-s)$ and $C(s)=C(-s)$, it turns out that $M_0(s)=M_\pi(-s)$ and hence $f_0(x)=f_\pi(-x)$ where $f_0(x)$ and $f_\pi(x)$ are the PDFs of $D$ in the cases $\theta_{ma}-\theta_{mb}=0$ and $\theta_{ma}-\theta_{mb}=\pi$ respectively. The EP is given by

$$p_e(a) = \frac{1}{2} \left( \int_{-\infty}^{0} f_0(x) dx + \int_{a}^{\infty} f_\pi(x) dx \right)$$

(33)

The optimum threshold is found by differentiating $p_e$ with respect to $a$, and setting the derivative equal to zero. The optimum threshold $a=a_{\text{opt}}$ that minimizes $p_e(a)$ is found to obey $f_0(a_{\text{opt}})-f_\pi(a_{\text{opt}})=0$, i.e. $f_0(x)$ and $f_\pi(x)$ intersect at the optimum threshold. Due to the symmetry of $f_0(x)$ and $f_\pi(x)$, illustrated in fig. 2, it is deduced that the optimum threshold is in fact $a_{\text{opt}}=0$. The minimum EP, $p_e$ is therefore given by

$$p_e = \int_{-\infty}^{0} f_0(x) dx = P_0(0)$$

(34)

where $P_0(y)=P(D<y|\theta_{ma}=\theta_{mb}=0)$ is the CDF of $D$ when $\theta_{ma}=\theta_{mb}=0$. To estimate $P_0(a)$, the saddle point approximation along with (31) can be used. As will be shown in the next section, in the absence of ASE noise, the EP can be evaluated in closed form, yielding the error floor set by the in-band crosstalk noise.

Although the present analysis assumes identical pulse shapes $g(t)$ for the signal and interfering components, it is useful to briefly sketch how the theory can be generalized to incorporate non-identical pulses shapes. If $g_m(t)$ are the pulse shapes of the signal ($m=0$) and the crosstalk components ($m>0$), one can expand $g_m(t)$ in terms of its Fourier coefficients $g_{mk}$ instead of $g_k$ as in (11). One can then derive an equation similar to (16) where the equations for $x_{ka}, y_{ka}, y_{kb}$ now involve the coefficient $g_{mk}$ instead of $g_k$. For example, $x_{ka}=\sum_m c_m g_{mk} e^{-im\theta}$. Equation (23) is valid provided the new values for $z=\{x_{1a}, \ldots, x_{ka}, x_{kb}, y_{ka}, y_{kb}, \ldots\}$ are taken into account. The variables $z$ are dependent Gaussian random variables with combined PDF given by $(2\pi)^{-M/2}S$
$\frac{1}{2} \exp(-\frac{1}{2} z^T G z)$ and $S$ and $G$ being the determinant and the inverse of the covariance matrix. To estimate the expectation of (23) one writes $< M_{DC}(s) >$ as $C(s) < \exp(z^T V(s) z) > = C(s) \int dz \exp(z^T K(s) z)$ where $K(s) = V(s) - \frac{1}{2} G$ is a Hermitian matrix. This integral can be in principle estimated by diagonalizing $K(s)$ and one can then evaluate $p_e$ by the saddle point approximation.

It is worth noting that when both the crosstalk and the ASE noises are present, then the total noise can behave quite differently than an additive white Gaussian noise. To show this, the simple case where $\sigma_k^2 = \sigma_{ase}^2$ and there are $L$ spectral ASE components inside the receiver’s bandwidth is considered. From (25) it can be ascertained that

$$A(s) = \frac{\sigma_{ave}^2 s / 2}{2(1 - \sigma_{ave}^4 s^2 / 4)} \quad (35a)$$

$$B(s) = \frac{s}{1 - \sigma_{ave}^4 s^2 / 4} \quad (35b)$$

$$C(s) = \left( \frac{1}{1 - \sigma_{ave}^4 s^2 / 4} \right)^L \quad (35c)$$

provided that the pulses are normalized so that $G_H = 1$ where $G_H$ is given by

$$G_H = \frac{1}{2} \sum_k |g_k|^2 = \frac{1}{2T} \int_{-\infty}^{\infty} |g_H(t)|^2 dt \quad (36)$$

The MGF of the decision variable in the case $\theta_{ma} - \theta_{mb} = 0$ becomes

$$M_D(s) = \frac{1}{1 - s^2 \sigma_t^4 / 4} \left( \frac{1}{1 - s^2 \sigma_{ase}^4 / 4} \right)^{L-1} \exp \left( \frac{c_2^2 s}{1 - s \sigma_t^2 / 2} \right) \quad (37)$$

where $\sigma_t^2 = \sigma_{ase}^2 + 2 \sigma^2$ and $L \geq 1$. In the case where no crosstalk noise exists ($\sigma_t^2 = 0$), and assuming that the ASE noise spectral components all have the same power $\sigma_{ase}^2 = p_{ase}^2$, then (37) reduces to

$$M_D^{ase}(s) = \left( \frac{1}{1 - s^2 p_{ase}^4 / 4} \right)^L \exp \left( \frac{c_2^2 s}{1 - s p_{ase}^2 / 4} \right) \quad (38)$$
which is the MGF of the preamplified DPSK receiver without any crosstalk noise.

There are some important conclusions that can be drawn from (37) and (38). In the case where \( L=1 \), the MGF of the decision variable \( M_D(s) \) in (37) becomes similar to in form to \( M_D^{ase}(s) \) in (38), provided that \( p_{ase}^2 \) is replaced with \( \sigma_t^2 \). This means that if \( L=1 \) the simple exponential formula \( \frac{1}{2} \exp(-\frac{c_0^2}{\sigma_t^2}) \) derived for the EP of the preamplified DPSK receiver (in the absence of the crosstalk noise).

The situation is different however if \( L>1 \). In this case there is no way to choose \( p_{ase}^2 \) so that (38) and (37) become similar. This means that the crosstalk noise cannot generally be incorporated in the preamplified receiver model by simply adjusting the value of the optical signal-to-noise ratio.

V. ERROR FLOOR DUE TO IN-BAND CROSSTALK NOISE

To estimate the error floor due to the in-band crosstalk noise, the ASE noise is neglected (i.e. \( \sigma_k=0 \)). The decision variable reduces \( D \) to

\[
D = G_{m}(X_a X_b + Y_a Y_b)
\]

i.e. in the absence of optical amplification and as \( M \rightarrow \infty \), \( D \) is the sum of two independent random variables \( X_a X_b \) and \( Y_a Y_b \) each of which is the product of two independent Gaussian random variables. To verify that \( D \) indeed has this asymptotic behavior, the behavior of both \( D \) and the auxiliary variables for finite \( M \) is considered.

The PDF of \( X_a \) is illustrated in Fig. 3, for \( \phi_0=0 \) and \( M=10,50 \) along with a Gaussian PDF with mean value \( m_{xa} \) and variance equal to \( \sigma \). The amplitude of the signal \( c_0 \) is taken equal to 10, while \( \sigma^2=4 \). The interfering components all have the same amplitude, i.e. for \( m>0 \), \( c_{m}=c_{i}=(2/M)^{1/4} \sigma \). To estimate the PDF of \( X_a \) for finite \( M \), the Multi-Canonical Monte Carlo (MCMC) method was used. As in conventional Monte-
Carlo sampling, the MCMC method generates samples of the random variable and estimates the PDF from the occurrences of the samples. The sample generation procedure consists of many iterations. In the first iteration, conventional Monte-Carlo sampling is performed to obtain an estimate for the PDF of $X_a$. The information gained is used in the next iteration to bias the samples and increase the occurrence of the values of $X_a$ at the tails of its PDF. The computed PDF is then used to further bias the samples and so on. This procedure allows the accurate computation of the PDF even at very low values, without an excessive number of samples. The details of this method can be found in [15]-[17]. Figure 3, clearly illustrates the asymptotic convergence of the PDF of $X_a$ to its asymptotic Gaussian form. Although for $M=10$ there is some difference between the actual and the asymptotic (Gaussian) PDFs of $X_a$, this difference is significantly reduced for $M=50$. A Gaussian assumption was made for the auxiliary variables used in the case of a direct detection ASK receiver in [10] as well. The two auxiliary variables in that case are similar to the ones defined here, except for the existence of $e_{ma}$ and $e_{mb}$, which are absent in the auxiliary random variables of [10]. One could expect that the presence of the discrete random variables $e_{ma}$ and $e_{mb}$ in the expressions may affect the convergence of the auxiliary variables $X_a$, $Y_a$, $X_b$ and $Y_b$. However, as shown in figure 3 for $M=50$, the PDF of these variables is approximated by its Gaussian asymptotic form quite well.

The convergence of $D$ to its asymptotic form, is illustrated in Fig. 4, where the MCMC method is used to obtain the PDF of $D$ in the case where $M=20$ and $M=50$. The amplitude of the signal and the crosstalk components are again $c_0=10$ and $c_m=c_1=(2/M)^{1/2}\sigma$ respectively, with $\sigma=2$. Also plotted in fig. 4, is the asymptotic PDF of $D$ obtained by the MCMC method, assuming that $X_a,X_b,Y_a$ and $Y_b$ are uncorrelated Gaussian random variables with mean values and standard deviations determined by
(14) and (15) respectively. It is observed that as expected, the PDF of $D$ for finite $M$, converges to the PDF of the sum of two independent random variables each of which is the product of two independent Gaussian random variables. Infact the PDF for $M=50$ is quite close to the asymptotic PDF and this suggests that the asymptotic model can be used with great accuracy if $M>50$.

As indicated by (34), the EP is related to the integral of the tails of the PDF of $D$ inside $(-\infty,0]$. As $M$ increases, the left tails of the PDF are moving to further to the left and hence for $M \to \infty$ the value of the EP can be higher than for small values of $M$, especially $M=1$ [11]. Hence the EP, calculated for $M \to \infty$ will provide an upper bound for the EP in the case of finite $M$. The fact that as the number of interferers $M$ is increased and the signal to crosstalk ratio $SX = c_0^2/(2\sigma^2)$ is kept constant, the EP increases, has been observed in the case of ASK modulation [7] as well.

The probability $P(D<0)$ for such a random variable can be estimated in a closed form as in [18]. Applying the results of [18] in the present case, the minimum EP is given by

$$p_e = \frac{1}{2} \exp\left(-\frac{c_0^2}{2\sigma^2}\right)$$

The MGF of the decision of $D$ in the absence of the ASE noise can be computed setting from (37) setting $\sigma_k^2 = \sigma_{ase}^2 = 0$. Figure 5, plots the results for the EP calculated by (40) are compared with the application of the saddle point approximation and (37). It is observed that both methods agree very well even for very low values of the EP. This result implies that the MGF (31), from which (37) is derived as a special case is indeed accurately estimated.

It is useful to compare the error floor of the DPSK and the ASK receivers due to the presence of in-band crosstalk. In the case of the ASK receiver, the decision variable $D$
is asymptotically written as the sum of the squares of two auxiliary random variables \( c_0B_0 + R \) and \( V \) [10], given by

\[
R = c_0B_0 + \sum_{m \geq 1} B_m c_m \cos \phi_m, \quad V = \sum_{m \geq 1} B_m c_m \sin \phi_m
\]  

(41)

where \( B_m (=0 \text{ or } 1) \) is the bit value of the signal (for \( m=0 \)) and of the \( m^{th} \) crosstalk component (for \( m>0 \)). It can be shown that the decision variable asymptotically becomes a chi-square random variable [10] with MGF given by

\[
M_{\text{ASK}}(s) = \frac{1}{1 - \sigma^2 s} \exp \left( \frac{c_0^2 B_0 s}{1 - \sigma^2 s} \right)
\]  

(42)

where \( \sigma^2 = \langle R^2 \rangle + \langle V^2 \rangle \) is again given by (27). Note, that in the case of ASK, unlike the DPSK receiver, no analytic result can be obtained for the error floor, since the PDFs of the decision variable for \( B_0=0 \) and \( B_0=1 \) are not symmetric. As a consequence, the optimum threshold position is not known a priori. One alternative is to apply the saddle point approximation in order to calculate the optimum threshold and the minimum EP numerically.

In Fig. 6, the value of the error floor due to in-band crosstalk is plotted for the ASK and the DPSK receivers for various values of the signal to crosstalk ratio \( SX = c_0^2 / (2\sigma^2) \). It is evident that the ASK receiver is more affected by the in-band crosstalk noise. In fact, for \( P_e = 10^{-9} \), the required signal to crosstalk ratio for the DPSK receiver \( \cong 13.1 \text{dB} \) and is about 3dB lower from that of the ASK receiver. This 3dB improvement is similar to the improvement encountered in the case when only the ASE noise is present [13]. Note that the value of \( SX = 13.1 \text{dB} \) for which \( P_e = 10^{-9} \) is higher than that of [11] because in [11] only one interfering component (\( M=1 \)) is assumed and as explained previously the EP for the same SX increases as \( M \) increases.
If all interferers have the same amplitude, then \( SX = \frac{c_0^2}{2\sigma^2} = X/M \) where \( X = \frac{c_0^2}{c_1^2} \). In the case of a passive AWG \( N_i \times N_i \) interconnection, the number of interfering components \( M \), equals \( N_i - 1 \) \cite{3}. Hence, if DPSK is employed, the receiver at each node can tolerate twice as much in-band crosstalk noise power and hence the maximum number of nodes \( N_i \) that can be interconnected is approximately doubled.

VI. INFLUENCE OF THE ASE NOISE.

For simplicity, it can be assumed that the pulses \( g_H(t) \) are normalized so that \( G_H = 1 \). If \( g_H(t) \) takes negligible values outside \([0, T]\), then \( G_H \) can be approximated by

\[
G_H \approx \frac{1}{2T} \int_{-\infty}^{\infty} |g_H(t)|^2 dt = \frac{1}{2T} \int_{-\infty}^{\infty} |G(f)|^2 |H(f)|^2 df
\]  

(43)

In (42), \( G(f) \) and \( H(f) \) are the Fourier transform of \( g(t) \) and the transfer function of the optical receiving filter respectively. The second equality holds as a consequence of Parseval’s identity \cite{14} and since \( G(f)H(f) \) is the Fourier transform of \( g_H(t) \). If \( G(f) \) is much narrower than \( H(f) \), then \( G(f)H(f) \approx G(f) \) (i.e. the optical filter does not significantly alter the pulse spectrum), then

\[
\frac{1}{2T} \int_0^T |g(t)|^2 dt \approx G_H = 1
\]  

(44)

which means that the normalization of \( g(t) \) is approximately the same as that of \( g_H(t) \).

The energy of the optical field \( S_0(t) \) in \([0, T]\) is given by \( \frac{1}{2} \int_0^T |S_0(t)|^2 dt \). Taking into account (43), it is easy to deduce that \( Tc_0^2 \) is the number of photons of the signal at the output of the optical amplifier. Similarly, \( Tc_m^2 \), for \( m > 0 \), is the number of photons of the \( m^{th} \) crosstalk component.

The influence of the ASE noise is illustrated in Fig. 7, where the EP of a DPSK receiver is plotted as a function of the input power \( P_{in} \) at the input of the optical amplifier. The signal to crosstalk ratio is assumed \( SX = 13.1 \) dB, which corresponds to
an error floor equal to $10^{-9}$. The optical amplifier is assumed to have $G=30\text{dB}$ and $n_{sp}=2$. The optical filter is assumed rectangular with bandwidth $W_o$ equal to 100GHz, i.e. $H(f)=1$ for $|f|\leq W_o/2$. Since $Tc_0^2$ is the number of photons of the signal inside the bit duration, and if for simplicity $g_H(t)$ is assumed rectangular inside $[0,T]$, then $P_{in}T/(hf)=Tc_0^2$ and $c_0^2=P_{in}/(hf)$. In the figure, $N$ denotes the number of optical amplifiers that the signal and the crosstalk components pass before reaching the receiver’s optical filter. The last amplifier is that of the pre-amplified receiver and in the case $N=1$ only this amplifier is assumed. Also plotted in the figure is the error floor of the crosstalk noise obtained using (40). For low input powers, the ASE noise is dominant, especially if $N>1$. As $P_{in}$ increases, the EP is improved because the ASE noise power is reduced compared to the signal power. However, the EP can not improve beyond $10^{-9}$ which is the error floor set by the in-band crosstalk noise.

In figure 8, the EP is plotted for various values of the signal-to-crosstalk ratio $SX$ for $N=5$. It is deduced that the EP is severely affected by the value of SX even in the presence of the ASE noise. Indeed if SX is increased by 1dB, from 13dB to 14dB, the EP can vary by about 2 orders of magnitude depending on the input power. It should also be noted that the in-band crosstalk noise does not simply set an error floor but can significantly affect the relation between the EP and the input power even if the EP is several orders larger than the error floor. This is evident in Fig 8, since the value of SX influences the rate in which the EP is decreased as $P_{in}$ increases. It is therefore verified that the crosstalk-noise can have important implications in the performance of the receiver even in the presence of ASE noise.

VII. CONCLUSIONS

In this paper, the influence of the in-band crosstalk in a DPSK was theoretically investigated. An expression for the MGF of the decision variable was derived which
takes into account the ASE noise of the optical amplifiers. This expression for the
MGF, along with the saddle point approximation provides a useful model for the
evaluation of the error probability in DPSK receivers in the presence of in-band
crosstalk noise. A closed form formula for the error floor set by the in-band crosstalk
noise was also given.

FIGURE CAPTIONS

Figure 1: A typical DPSK receiver diagram.

Figure 2: Convergence of the PDF of $X_a$ to its asymptotic Gaussian form as the
number of interferers $M$ increases.

Figure 3: Convergence of the PDF of $D$ to its asymptotic form as the number of
interferers $M$ increases.

Figure 4: Symmetry of probability density function of the decision variable of the
DPSK receiver limited by in-band crosstalk noise.

Figure 5: Comparison of the results obtained by the saddle point method (rectangles)
and equation (33) (solid line).

Figure 6: Error probabilities for an ASK (solid line) and a DPSK (dashed line)
receiver for various values of the signal to crosstalk ratio $c_0^2/(2\sigma^2)$

Figure 7: Influence of the ASE in the EP of a DPSK receiver. $N$ denotes the number
of amplifiers that the signal passes.

Figure 8: The EP of a preamplified DPSK receiver for three different values of the
signal-to-crosstalk ratio SX.
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
$c_0^2/(2\sigma^2)$ (dB)

Figure 6
Figure 7
Figure 8
REFERENCES


